Quasi-projective characters in a block

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Burkhard Külshammer's 60th birthday, Jena, July 22 -25, 2015

Essen, 198?

Marseille, in front of Notre Dame de la Garde, 1986?

1. Introduction (A. Zaleski)

Def. a) An ordinary character Λ of a finite group G is called quasi-projective if

$$\Lambda = \sum_{\varphi \in \mathsf{IBr}_p(G)} a_\varphi \Phi_\varphi \text{ with } a_\varphi \in \mathbb{Z}$$

where Φ_φ denotes the ordinary character associated to the projective cover of the module afforded by φ.

b) A p-Brauer character Φ is called quasi-projective if

$$\Phi = (\sum_{\varphi \in \mathsf{IBr}_p(G)} a_\varphi \Phi_\varphi)^\circ.$$

Def. We call a quasi-projective character Λ (resp. Φ) indecomposable if there is no splitting

$$\Lambda = \Lambda_1 + \Lambda_2 \qquad (\text{resp.} \Phi = \Phi_1 + \Phi_2)$$

with $\Lambda_i\,(\text{resp.}\Phi_i) \neq 0$ and quasi-projective character.

Remark. An indecomposable quasi-projective character belongs to a block.

To be brief we put

$\mathrm{Iqp}(B)$ = set of indecomposable quasi-projective ordinary characters of the p-block B

(call that: Hilbert basis for the decomp. matrix of B)

$\mathrm{IBqp}(B)$ = set of indecomposable quasi-projective Brauer characters of B.

(call that: Hilbert basis for the Cartan matrix of B)

Example.

$G = A_5$, $p = 2$, B_0 the principal block.

$\mathrm{Irr}(B_0) = \{\chi_1, \chi_2, \chi_3, \chi_5\}$ degrees: $1, 3, 3, 5$

$\mathrm{IBr}_2(B_0) = \{\beta_1, \beta_2, \beta_3\}$ degrees: $1, 2, 2$

$|\mathrm{Iqp}(B_0)| = 4$:

$\Phi_1 - \Phi_3 = 1 + \chi_2,$

$\Phi_1 - \Phi_2 = 1 + \chi_3,$

$\Phi_2 = \chi_2 + \chi_5,$

$\Phi_3 = \chi_3 + \chi_5$

$|\mathrm{IBqp}(B_0)| = 6$:

$(3\Phi_1 - 2\Phi_2 - 2\Phi_3)^\circ = 4\beta_1$

$(2\Phi_2 - \Phi - 1)^\circ = 2\beta_2$

$(2\Phi_3 - \Phi_1)^\circ = 2\beta_3$

$(\Phi_1 - \Phi_3)^\circ = 2\beta_1 + \beta_2$

$(\Phi_1 - \Phi_3)^\circ = 2\beta_1 + \beta_3$

$(\Phi_2 + \Phi_3 - \Phi_1)^\circ = \beta_2 + \beta_3$

Example.

$G = PSL(2,7)$, $p = 7$, B_0 the principal block.

$\mathrm{Irr}(B_0) = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$ degrees: $1, 3, 3, 6, 8$

$\mathrm{IBr}_7(B_0) = \{\beta_1, \beta_2, \beta_3\}$ degrees: $1, 3, 5$

$|\mathrm{Iqp}(B_0)| = 5$:

$1 + \chi_4$, $1 + \chi_2 + \chi_3$, $\chi_4 + \chi_5$, $\chi_2 + \chi_3 + \chi_5$

$|\mathrm{IBqp}(B_0)| = 11$:

$7\beta_1$, $7\beta_2$, $7\beta_3$, $\beta_1 + 4\beta_3$, $\beta_2 + 5\beta_3$, $4\beta_1 + \beta_2$, $\beta_1 + \beta_2 + 2\beta_3$, $2\beta_1 + \beta_3$, $2\beta_2 + 3\beta_3$, $\beta_1 + 2\beta_2$, $3\beta_2 + \beta_3$

Example.

$G = McL$, $p = 2$, B_0 the principal block

$|\mathrm{Irr}(B_0)| = 18, \quad |\mathrm{IBr}_2(B_0) = 8$

$|\mathrm{Iqp}(B_0)| = 38 = 2.19$

$|\mathrm{IBqp}(B_0)| = 8304 = 2^4.3.173$

Problems.

1. What is the meaning of (indecomposable) quasi-projective?
2. What can we say about $\mathrm{Iqp}(B)$ or $\mathrm{IBqp}(B)$?
3. Is there a reasonable good function in terms of B which bounds $|\mathrm{Iqp}(B)|$ or $|\mathrm{IBqp}(B)|$?

2. Hilbert bases

Let D, C denote the decomposition resp.Cartan matrix of a block B.

Quasi-projective characters

$$\sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(B)} \underbrace{\Big(\sum_{\varphi \in \mathrm{IBr}(B)} d_{\chi\varphi} a_\varphi \Big)}_{=(Da)_\chi \geq 0} \chi.$$

$$\Big(\sum_{\varphi \in \mathrm{IBr}(B)} a_\varphi \Phi_\varphi \Big)^\circ = \sum_{\psi \in \mathrm{IBr}(B)} \underbrace{\Big(\sum_{\varphi \in \mathrm{IBr}(B)} c_{\psi\varphi} a_\varphi \Big)}_{=(Ca)_\psi \geq 0} \psi.$$

Hilbert basis of a matrix $A \in (\mathbb{Z})_{k,l}$

Def. $cone(A) = \{x \in \mathbb{R}^l \mid Ax \geq 0\}$

Facts.

a) (Gordon 1873, Hilbert 1890)

$cone(A)$ is generated by a finite so-called integral Hilbert-basis; i.e.,

$\exists\ h_1, \ldots, h_t \in cone(A) \cap \mathbb{Z}^l$ s.t. any $c \in cone(A) \cap \mathbb{Z}^l$ can be written as $c = \sum_{i=1}^{t} a_i h_i$ with $a_i \in \mathbb{N}_0$.

b) (van der Corput, 1931)

If ker$A = 0$, then a minimal integral Hilbertbasis is unique. Denote them by $\mathcal{H}_A$.

c) If ker$A = 0$, then $A\mathcal{H}_A$ are the indecomposable vectors in $A(cone(A) \cap \mathbb{Z}^l)$.

Applications.

- $$D\mathcal{H}_D = \mathrm{Iqp}(B)$$

- $$C\mathcal{H}_C = \mathrm{IBqp}(B)$$

Explicit computations. Software package 4ti2 (Hemmecke, Köppe, Malkin, Walter)

3. Indecomp. quasi-projective ordinary characters.

- quasi-projective character = p-vanishing character
- $|G|_p \mid \Lambda(1)$ if Λ is quasi-projective
- $\chi \in \mathrm{Irr}(B)$ quasi-projective $\Rightarrow$ B of defect zero
- $|\mathrm{Iqp}(B)| \geq |\mathrm{IBr}(B)|$

Def. A Brauer character φ is called quasi-liftable if there exists an ordinary character χ such that $\chi^\circ = b\varphi$ with $b \in \mathbb{N}$

Lemma. (cf. Navarro, 10.16) If $\Lambda = \sum_\varphi a_\varphi \Phi_\varphi$ is a quasi-projective character and φ is quasi-liftable, then $a_\varphi \geq 0$.

(If $\chi^\circ = n\varphi$, then $na_\varphi = (\Lambda, n\varphi)^\circ = (\Lambda, \chi) \geq 0$.)

Example. $G = {}^2F_4(2)'\dot{2}$ and $p = 2$.

There exists a non-liftable $\beta \in \mathrm{IBr}(G)$ and $\chi, \psi \in \mathrm{Irr}(G)$ such that $\chi^\circ = 2\beta$ and $\psi^\circ = 3\beta$.

Theorem. Equivalent are:

a) $\mathrm{Iqp}(B) = \{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(B)\}$.

b) Every $\beta \in \mathrm{IBr}(B)$ is quasi-liftable.

Proof. b) $\Rightarrow$ a) Navarro's Lemma or

- $$D = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & n_l \\ & * & \end{pmatrix}$$

- $Da \geq 0 \ (a \in \mathbb{Z}^l) \ \Rightarrow \ a \geq 0$

a) $\Rightarrow$ b) Suppose that $\beta \in \mathrm{IBr}(B)$ is not quasi-liftable.

- For each $\chi \in Irr(B)$ with $d_{\chi,\beta} \neq 0$ there exists a $\beta \neq \psi \in \mathrm{IBr}(B)$ with $d_{\chi,\psi} \neq 0$.

- $b = \max\{d_{\chi,\beta} \mid \chi \in \mathrm{Irr}(B)\}$

- $\Lambda = -\Phi_\beta + b \sum_{\varphi \neq \beta} \Phi_\varphi$

- $(\Lambda, \chi) = -d_{\chi,\beta} + b \sum_{\varphi \neq \beta} d_{\chi,\varphi} \geq 0$

- Λ quasi-projective, not projective character.

Question.

Are the following equivalent?

a) $\mathrm{Iqp}(B) = \{\Phi_\varphi \mid \varphi \in \mathrm{IBr}_p(B)\}$.

b) Each $\varphi \in \mathrm{IBr}_p(B)$ is quasi-liftable.

c) Each $\varphi \in \mathrm{IBr}_p(B)$ is liftable.

Theorem.

Let B be a block with a cyclic defect group > 1. By χ_0 we denote the sum of exceptional irreducible characters of B (if such characters exist). Furthermore let $\mathrm{Irr}^0(B)$ be the set consisting of χ_0 and all the non-exceptional irreducible characters of B. Then

$$\Lambda = \sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi_\varphi \in \mathrm{Iqp}(B)$$

if and only if $\Lambda = \chi + \psi$ for $\chi, \psi \in \mathrm{Irr}^0(B)$ where the distance between χ and ψ in the Brauer tree is odd.

Example.

$G = PSL(2, 17)$, $p = 17$, B_0 the principal 17-block

$$20 = |\mathrm{Iqp}(B_0)| \not\leq |\delta(B_0)| = 17$$

Question.

How to bound $|\mathrm{Iqp}(B)|$ in terms of invariants of B?

- If all $\varphi \in \mathrm{IBr}(B)$ are quasi-liftable, then $\mathrm{Iqp}(B) = \{\Phi_\varphi | \varphi \in \mathrm{IBr}(B)\}$.

- $l(B) = 1 \;\Rightarrow\; \mathrm{Iqp}(B) = \{\Phi_\varphi\}$

- Does $l(B) = 2$ imply $|\mathrm{Iqp}(B)| = 2$?
$(G = 2.A_8, p = 3, \text{block } \#5)$

Let $\Lambda = \sum_{\varphi \in \mathsf{IBr}(B)} a_\varphi \Phi_\varphi = \sum_{\chi \in \mathsf{Irr}(B)} b_\chi \chi \in \mathsf{Iqp}(B)$.

Minkowski 1896:

$$cone(D) = cone(\{a^1, \dots, a^m \mid 0 \neq a^i \in \mathbb{Z}^d\}),$$
$$\text{where } m \leq \binom{k}{l-1}$$

- a^i are solutions of $l-1$ linearily independent equations of $Dx = 0$.

- $\mathcal{H}_D \subseteq \{a^1, \dots a^m\} \cup$
$$\{a \in cone(D) \cap \mathbb{Z}^d \mid a = \textstyle\sum_i \lambda_i a^i, \lambda_i \in [0,1)\}$$

Ewald/Wessel '91:

- If $l \geq 2$, then

$$|a_\varphi| \leq (l-1) \max_i \| a^i \|_\infty$$

4. Indecomp. quasi-projective Brauer characters

Theorem. Let $d(B)$ denote the defect of the block B.

a) For each $\varphi \in \mathrm{IBr}_p(B)$ there is a minimal p-power, say $p^{a(\varphi)}$ such that $p^{a(\varphi)}\varphi \in \mathrm{IBqp}(B)$ where $a(\varphi) \leq d(B)$.

b) There exists $\varphi \in \mathrm{IBr}_p(B)$ with $a(\varphi) = d(B)$

Consequence. If $\varphi \in \mathrm{IBr}(B)$, then

$$\overline{\varphi(x)} = \begin{cases} p^{a(\varphi)}\varphi(x) & \text{for } x \text{ a } p\text{'-element,} \\ 0 & \text{otherwise.} \end{cases}$$

is a generalized character of B.

Example. $G = A_5$, $p = 2$, B_0 the principal block

elementary divisors: 4,1,1

$2^{a(\varphi)}$ for $\varphi \in \mathrm{IBr}(B_0)$: 4,2,2

Question. Does $a(\varphi) = 0$ for $\varphi \in \mathrm{IBr}(B)$ imply that B is of defect 0?

Question. Can one characterize blocks B with $|\mathrm{IBqp}(B)| = |\mathrm{IBr}(B)|$?

Fact. We always have $a(\varphi) \geq d(B) - ht(\varphi)$ where $ht(\varphi) = \nu_p(\varphi(1)) - \nu_p(|G|) + d(B)$.

Question. Is

$$a(\varphi) = d(B) - ht(\varphi),$$

if G is p-solvable and $\varphi \in \mathrm{IBr}(B)$?

Example. $G = McL$, $p = 2$, $\varphi \in \mathrm{IBr}(B_0)$ of degree 7.2^9.

- $|G|_2 = 2^7$
- $a(\varphi) = |d(B) - ht(\varphi)| = 2$

Happy Birthday