It’s a long way to the land of endo-trivial modules

Jacques Thévenaz

EPFL, Lausanne, Switzerland

24 July 2015

- G finite group, k algebraically closed field of characteristic p. Assume that p divides $|G|$.
- (Alperin 1977, Dade 1978). A kG-module M is endo-trivial if

$$\mathrm{End}_k(M) \cong k_G \oplus (\mathrm{proj}) , \quad \text{that is,} \quad M \otimes_k M^* \cong k_G \oplus (\mathrm{proj})$$

in other words

$$M \otimes M^* \cong k_G \quad \text{in the stable category } \mathbf{stmod}(kG) .$$

Example

$\dim(M) = 1 \quad \Longrightarrow \quad M$ is endo-trivial.

Example

$\Omega^n(k_G)$ is endo-trivial.
In particular, the augmentation ideal $\Omega^1(k_G) = \mathrm{Ker}(kG \to k)$ is endo-trivial.

- Define

$$T(G) := \{ \text{ iso. classes of endo-trivial } kG\text{-modules in } \mathbf{stmod}(kG)\}$$

Each class $[M]$ contains an indecomposable kG-module M_0 (unique up to isomorphism), that is, $M \cong M_0 \oplus (\text{proj})$.

- $T(G)$ is an abelian group :

$$[M] \cdot [N] = [M \otimes N] \,, \qquad [k_G] = \text{ identity } , \qquad [M]^{-1} = [M^*].$$

For instance, $[\Omega^n(k_G)] \cdot [\Omega^m(k_G)] = [\Omega^{n+m}(k_G)]$.

- There is an obvious finite subgroup $X(G) := \{1\text{-dimensional } kG\text{-modules }\} \cong \mathrm{Hom}(G, k^\times) \cong (G/G')^*$, where G/G' is the largest quotient of G which is abelian and p'.
- If a Sylow p-subgroup is neither cyclic nor generalized quaternion, then $[\Omega^1(k_G)]$ generates an infinite cyclic subgroup, that is, $\cong \mathbb{Z}$.

Our main problem is :

Question

Classify all indecomposable endo-trivial kG-modules.
Equivalently, describe the abelian group $T(G)$.

Theorem

(Dade, 1978). If G is an abelian p-group, then $T(G)$ is cyclic generated by $[\Omega^1(k_G)]$.

This theorem opened a completely new area of representation theory (Dade's lemma).

Theorem

(Puig, 1980, published in 1990, Carlson-Mazza-Nakano, 2006).
$T(G)$ is a finitely generated abelian group.

Theorem

$T(G)$ is a finitely generated abelian group.

Therefore $T(G) = TT(G) \oplus TF(G)$,
where $TT(G) =$ finite abelian group and $TF(G) \cong \mathbb{Z}^N$.

- The integer N is known, by Alperin (2001) for p-groups, Carlson-Mazza-Nakano (2006) in general.
- $N = 1$ in most cases, with $TF(G) \cong \mathbb{Z}$ generated by $[\Omega^1(k_G)]$.
- Thus $TF(G)$ is essentially understood, except for explicit generators in some cases.

But there is still an open problem :

Question

Can we find generators for $TF(G)$ which all lie in the principal block ?

Question

$TT(G) = ?$

Theorem

(Carlson-Thévenaz, 2005). If P is a p-group, then $TT(P) = \{1\}$, except if P is cyclic, generalized quaternion, or semi-dihedral.

- This uses heavy cohomological machinery.
- Puig's theorem follows as an easy corollary (but this proof is in fact much harder than Puig's original approach).
- $T(P) \cong \mathbb{Z}/2\mathbb{Z}$, if P is cyclic of order ≥ 3.
- $T(P) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, if P is generalized quaternion.
  $\exists$ an exotic endo-trivial module of order 2.
- $T(P) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, if P is semi-dihedral.
  $\exists$ an exotic endo-trivial module of order 2.

Our next problem is now :

Question

How to pass from P to G, where $P \in \mathrm{Syl}_p(G)$?

Define

$$\color{red}{K(G)} := \mathrm{Ker}\left(\mathrm{Res} : T(G) \longrightarrow T(P)\right).$$

- $K(G)$ is finite (because finitely many trivial source modules).
- $K(G) = TT(G)$, except if P is cyclic, generalized quaternion, or semi-dihedral.
- Often, $K(G) = X(G)$, but there are many interesting exceptions.
- Since 2005, many results on $TT(G)$ for various groups (p-solvable, symmetric, Lie type, sporadic, cyclic Sylow, generalized quaternion Sylow, dihedral Sylow, semi-dihedral Sylow, etc.).
- Work of Carlson, Hemmer, Koshitani, Lassueur, Malle, Mazza, Nakano, Navarro, Robinson, Schulte, Thévenaz, etc.

Question

Today, what are the new paths to our beloved land ?

- **New path 1 : Search for *simple* endo-trivial modules.**

Theorem

(Robinson, 2011). Let V be a simple endo-trivial kG-module. Then
• either V is simple endo-trivial on restriction to a quasi-simple normal subgroup L (actually $F^*(G) = Z(G)L$),
• or V is induced from a 1-dimensional module of a strongly p-embedded subgroup.

- Hence one can start investigating every quasi-simple group.

Theorem

(Lassueur-Malle, 2015). Let G be a finite quasi-simple group having a faithful simple endo-trivial module. Then the p-rank of G is at most 2.

- **New path 2 : Character-theoretic methods.**

Theorem

(Alperin, 2001, Lassueur-Malle-Schulte, 2013). Let $(K, \mathcal{O}, k)$ be a p-modular system and let M be an endo-trivial kG-module. Then M can be lifted to an endo-trivial $\mathcal{O}G$-module $\widehat{M}$.

- Hence one can start investigating the ordinary character $\chi_{\widehat{M}}$.

Theorem

(Lassueur-Malle, 2015). Let M be an indecomposable trivial source kG-module, i.e. $[M] \in K(G)$. Let $\widehat{M}$ be its lift to $\mathcal{O}$.

$$M \text{ is endo-trivial} \iff \chi_{\widehat{M}}(x) = 1\,,\ \forall \text{ non-trivial } p\text{-element } x \in G\,.$$

- Hence an extremely efficient way of finding exceptional endo-trivial modules. Read the ATLAS !
- Ordinary characters are much easier than *kG*-modules !

- **New path 3 : Balmer's weak homomorphisms.**
- Let P be a Sylow p-subgroup of G.
  A map $\psi : G \to k^\times$ is called a *weak P-homomorphism* if

$$\begin{array}{ll} \psi(u) = 1 \;\text{ if } u \in P\,, & \psi(g) = 1 \;\text{ if } P \cap P^g = \{1\}\,, \\ \psi(ab) = \psi(a)\psi(b) & \text{if } p \text{ divides } P \cap P^b \cap P^{ab}\,. \end{array}$$

- Let $A(G)$ be the abelian group of all weak P-homomorphisms.

Theorem

(Balmer, 2013). $K(G) \cong A(G)$.

- The proof is a journey in another land : tensor-triangulated categories.
- Hence one can start using $A(G)$ instead of $K(G)$.
- Weak P-homomorphisms are much easier than kG-modules ! In particular for computational purposes.

- Using weak homomorphisms, we obtain :

Theorem

(Carlson-Thévenaz, 2015). Suppose that P is abelian. Then

$$K(G) \cong \Big(N_G(P)/\rho^\infty(P)\Big)^* \cong N_G(P)/\rho^\infty(P)$$

for some specific subgroup $\rho^\infty(P)$ containing $N_G(P)'$.

- Hence an efficient way of finding exceptional torsion endo-trivial modules when P is abelian.
- Define inductively, for all nontrivial $Q \subseteq P$,

$$\begin{aligned}
&\rho^1(Q) = N_G(Q)' \quad \text{(where } N_G(Q)]/N_G(Q)' \text{ abelian and } p')\,,\\
&\rho^j(Q) = \langle\, N_G(Q) \cap \rho^{j-1}(R) \mid \{1\} \neq R \subseteq P \,\rangle\,,\\
&Q \subseteq N_G(Q)' = \rho^1(Q) \subseteq \ldots \subseteq \rho^m(Q) = \rho^{m+1}(Q) \subseteq N_G(Q)\,,\\
&\rho^\infty(Q) = \rho^m(Q) \;\; \text{when the increasing sequence stabilizes}\,.
\end{aligned}$$

- More generally, for an arbitrary Sylow p-subgroup P :

Question

Is $K(G)$ always isomorphic to $N_G(P)/\rho^\infty(P)$?

- Computational evidence.
- Theoretical difficulties for dealing with $N_G(Q)/\rho^\infty(Q)$ for all nontrivial $Q \subseteq P$.
- However there is some remarkable behavior :

Theorem

(Carlson-Thévenaz, work in progress). Let P be a Sylow p-subgroup of G.
For every nontrivial $Q \subseteq P$, there is a canonical embedding

$$N_G(Q)/\rho^\infty(Q) \longrightarrow N_G(P)/\rho^\infty(P)\,.$$

This embedding is an isomorphism if Q is characteristic in P.

The land of endo-trivial modules is not yet fully explored !

Thank you for your attention !