Symmetric bilinear forms in characteristic 2

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Outline

Vertices and Sources

- In this talk G is a finite group, k is an algebraically closed field of characteristic p and $p = 2$ unless stated otherwise.
- Let M be an indecomposable kG-module with vertex V and V-source Z.

So V is a p-subgroup of G, Z is a kV-module, $M \mid \mathrm{Ind}_V^G(Z)$, $Z \mid \mathrm{Res}_V^G(M)$ and V is minimal subject to the existence of Z.

- The trivial kG-module k_G has Sylow p-vertices and trivial sources.
- M is projective if and only if $V = 1$.

Duality and Bilinear Forms

- The dual of M is the kG-module $M^* = \mathrm{Hom}_k(M, k)$.
- There is a k-isomorphism between $\mathrm{Hom}_{kG}(M, M^*)$ and the space of G-invariant bilinear forms $b : M \times M \to k$.

Lemma (Gow, Willems)

If $p \neq 2$ each self-dual M is either symmetric or alternating. Moreover, the Krull-Schmidt theorem holds for symmetric/alternating modules.

If $p = 2$, alternating$\equiv$symmetric and symplectic $\Longrightarrow$ symmetric.
The Krull-Schmidt theorem fails for symmetric modules.

Lemma (Fong)

If $p = 2$, every non-trivial self-dual simple kG-is symplectic.

Induction and Restriction

- Let (M, b_M) be a symmetric kG-module and let (L, b_L) be a symmetric kH-module, where $H \leq G$.
- Recall that $\mathrm{Ind}_H^G(L) = \bigoplus_{gH} g \otimes L$.

$\mathrm{Res}_H^G(M, b_M)$ is the restricted symmetric kH-module and $\mathrm{Ind}_H^G(L, b_L)$ is the induced symmetric kG-module.

Permutation Modules

For $H \leq G$, and b_H a symmetric form on k_H, the cosets gH form an orthonormal basis for $\mathrm{Ind}_H^G(k_H, b_H)$.

Extended Defect Groups of Real 2-blocks

- $kG = B_0 \oplus \cdots \oplus B_n$ where each B_i is a p-block of G.
- B_i has a defect group $D_i \leq G$, determined up to G-conjugacy.
- $D_i = 1$ iff $B_i = \mathrm{End}_k(M_i)$ for some simple projective kG-module M_i.

Theorem (Gow)

If $p = 2$, each real B_i also has an extended defect group $E_i \leq G$. Then $[E_i : D_i] \leq 2$ and E_i is determined up to G-conjugacy.

Principal 2-block

The principal 2-block contains k_G. Its defect groups and extended defect groups are the Sylow 2-subgroups of G.

PIMs and Involutions

- Let e be a primitive idempotent in kG. So $P = kGe$ is a pim.
- If $t \in G$ is an involution, there is a symplectic bilinear form b_t on kG which has symplectic basis $\{g, gt\}_{g \in G}$.
- Let ^{o} be the contragredient map on kG; $g^o = g^{-1}$ for $g \in G$.

Theorem (Gow-Willems)

P has a symplectic form if and only if $e^t = e^o$, for some involution t. This occurs if and only if b_t is non-degenerate on P.

Its easy to see that $(kG, b_t) = \mathrm{Ind}_{\langle t \rangle}^{G}(k\langle t \rangle, b_t)$.

Theorem (M. 2009)

If P is simple and b_t is non-degenerate on P then P belongs to a real 2-block with defect group 1 and extended defect group $\langle t \rangle$.

So t is determined up to G-conjugacy; $\langle t \rangle$ is a 'symmetric vertex' of P.

B. Külshammer

Do other symmetric modules have symmetric vertices?

Symmetric Vertices

- Suppose that M, P are of symmetric type.

Definition

$T \leq G$ is a symmetric vertex of M if T is minimal such that $(M, b_M) \mid \mathrm{Ind}_T^G(L, b_L)$ for some b_M and symmetric kT-module (L, b_L).

Theorem (M. 2014)

A symmetric vertex of M contains a vertex of M with index at most 2.

- k_G has Sylow 2-symmetric vertices; $P(k_G)$ has symmetric vertex 1.
- If $P \neq P(k_G)$ then each symmetric vertex of P has order 2.

Are symmetric vertices unique up to G-conjugacy?

Theorem (M. 2014)

Let T be a symmetric vertex of M and let b_M be a T-projective symmetric form on M. Then b_M is H-projective if and only if $T \leq_G H$.

Example (Symmetric vertices are not unique)

Let S be the unique 2-dimensional simple kD_{12}-module.
Each non-central $C_2 \leq D_{12}$ is a symmetric vertex of $P(S)$.

- Let S be a self-dual simple kG-module, with symplectic form b_S.
- Let V be a vertex, let Z be a V-source and let T be a symmetric vertex of S which contains V.

Theorem (M., 2014)

The symmetric vertices of S are determined up to G-conjugacy and exactly one of the following holds:

(i) $T = V$ *and and* $Z \cong Z^*$.

(ii) $[T : V] = 2$ *and* $Z \cong Z^*$.

(iii) $[T : V] = 2$ *and* $Z \ncong Z^*$.

Multiplicity Modules

- $\mathrm{End}_k(S)$ is a primitive G-algebra.
- $\mathrm{End}_{kV}(S)$ has a unique maximal ideal $\mathcal{M}$ corresponding to Z.
- $\mathrm{End}_{kV}(S)/\mathcal{M} = \mathrm{End}_k(M_S)$ where the multiplicity module M_S is a module for a twisted group algebra $k_{\#}N_G(V,Z)/V$.

The adjoint σ of b_S is an involution on $\mathrm{End}_{kH}(S)$ for all $H \leq G$.
Now $\mathcal{M}^\sigma$ is the maximal ideal of $\mathrm{End}_{kV}(S)$ corresponding to Z^*. So

- σ is an involution on $\mathrm{End}_k(M_S)$ if $Z \cong Z^*$ or
- σ is an involution on $\mathrm{End}_k(M_S) \times \mathrm{End}_k(M_S^*)$ if $Z \ncong Z^*$.

Knörr

M_S is a simple projective $k_{\#}N_G(V,Z)/V$-module.

Case (i) $V = T$

- This occurs iff b_S is non-degenerate on some V-source Z.
- M_S lifts to the projective cover of the trivial $kN_G(V, Z)/V$-module.

Theorem (M. 2014)

$N_G(V, Z)/V$ has odd order and S belongs to the principal 2-block of G, S has multiplicity 1 as a component of $\mathrm{Ind}_V^G(Z)$ and S is the only non-degenerate component of $\mathrm{Ind}_V^G(Z, b_Z)$ that has vertex V.

- Note that if $Z \cong k_V$ then b_S is non-degenerate on Z iff $S \cong k_G$.

Case (i) Examples

Example

S a self-dual simple kG-module with Sylow 2-vertices e.g. $S = k_G$.

Example (Müller, Zimmermann)

For $n \geq 3$ the natural kS_{2n}-module $D^{[2n-1,1]}$ has vertex $V \in \mathrm{Syl}_2(S_{2n})$ and V-source $\mathrm{Res}_V^{S_{2n}}(S)$.

Example (Danz, Külshammer, Zimmermann)

The kS_5-module $D^{[3,2]}$ has Klein-four vertex $V \leq A_4$ and $\mathrm{Res}_V^{S_5}(D^{[3,2]}) = Z_1 \perp Z_2$ with $Z_i \cong Z_i^*$ but $Z_1 \not\cong Z_2$.

Case (ii) $[T : V] = 2$ and $Z \cong Z^*$

- b_S is degenerate on Z. So Z has even multiplicity in S_V.
- M_S lifts to a self-dual simple projective $kN_G(V, Z)/V$-module which has symmetric vertex T/V.

Corollary

If S is a self-dual simple kG-module which has a $N_G(V)$-invariant V-source then the multiplicity module M_S is an Alperin weight of G.

Case (ii) Examples

Example

S a self-dual non-trivial simple kG-module with trivial sources.

Example (Danz, Külshammer, Zimmermann)

The kS_7-module $D^{[4,3]}$ shares a vertex and sources with $D^{[3,2]}$.
$\mathrm{Res}_V^{S_7}(D^{[4,3]}) = (Z_1 \oplus Z_1) \perp (Z_2 \oplus Z_2)$ with each Z_i degenerate.

Example (M., Navarro)

G a solvable group and S is a non-trivial simple kG-module with self-dual sources.

Case (iii) $Z \ncong Z^*$

- M_S need not lift to a $kN_G(V,Z)/V$-module.
- Let $N_G^*(V,Z)$ be the stabilizer of $\{Z, Z^*\}$ in $N_G(V)$.
- There is an 'extended' multiplicity module for $k_\# N_G^*(V,Z)/V$.

Theorem (M. 2014)

$Z^T \cong Z^*$. In particular $N_G^*(V,Z)/V$ splits over $N_G(V,Z)/V$ and T/V is a complement to $N_G(V,Z)/V$ in $N_G^*(V,Z)/V$.

Case (iii) Examples

Example (Külshammer, Zimmermann, Robinson)

Let $H = \mathrm{GL}(n, 2)$, $n \geq 3$. Set $G = H\langle\tau\rangle$ where τ is transpose inverse. The natural kH-module induces a self-dual simple kG-module S. Each vertex of S is in $\mathrm{Syl}_2(H)$ but no source of S is self-dual.

For example when $n = 4$, $H = A_8$ and $G = S_8$. Then S is the 8-dimensional spin module $D^{[5,3]}$. Its sources are 4-dimensional.

Example (Dade)

Let $G = 3^{1+2} : \mathrm{SD}_{16}$ and let S be a self-dual simple kG-module, with $\dim(S) = 6$. Then S has vertex Q_8 and non self-dual sources.

Construct 3^{1+2} from a symplectic form on $\mathbb{F}_3^2$. Then SD_{16} is a Sylow 2 of $\mathrm{GL}(2, 3)$ and $\mathrm{Q}_8 = \mathrm{SD}_{16} \cap \mathrm{Sp}(2, 3)$. Let N be a simple 3-dimensional $k3^{1+2} : \mathrm{Q}_8$-module. Then $S = N^G$ and N_{Q_8} is a Q_8-source of S.

Solvable Groups

For the remainder of the talk G is a finite solvable group.
We discuss some joint work with G. Navarro.

Theorem (Fong-Swan)

Every irreducible Brauer character of G can be lifted to an ordinary irreducible character of G.

Nucleus

- Let S be a simple kG-module with Brauer character θ.
- Let $\chi \in \mathrm{Irr}(G)$ be a 'lift' of θ.

Lemma

There exists $W \leq G$ and $\eta \in \mathrm{Irr}(W)$ such that $p \nmid \eta(1)$ and $\eta^G = \chi$.

- *The Sylow p-subgroups of W are determined up to G-conjugacy as they are vertices of S.*

Example

Let $\chi \in \mathrm{B}_{p'}(G)$ be an Isaacs lift of θ and (W, η) an Isaacs nucleus of χ. Then η is a p'-special character in the sense of Gajendragadkar.

Extended Nucleus

- From now on $p = 2$ and S is non-trivial and self-dual.
- Let θ and χ be as above.

Theorem (M., Navarro 2015)

There exists $U \leq G$ and $\delta \in \mathrm{Irr}(U)$ with δ real, $\delta^G = \chi$ and $\delta(1)/2$ odd.

- *The Sylow 2-subgroups of U are determined up to G-conjugacy as they are symmetric vertices of S.*

Example

Let $\chi \in \mathrm{B}_{2'}(G)$ be an Isaacs lift of θ with Isaacs nucleus (W, η).
Then $\eta \neq \overline{\eta}$ and we can take $U = N_G^*(W, \eta)$ and $\delta = \eta^U$.

Solution to a question

- $\nu(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$ the Frobenius-Schur indicator of χ.
- $\nu(\chi)$ indicates if χ is non-real or real-valued and the character of a real or non-real representation.

W. Willems asked if each real-valued irreducible Brauer character of a solvable group can be lifted to the character of a real representation.

Theorem (Isaacs (?), M., Navarro 2015)

Let χ be the Isaacs canonical lift of θ. Then $\nu(\chi) = +1$.

Example

$G = C_3 : C_4$ non-abelian, $\theta \in \mathrm{IBr}_2(G)$ with $\theta(1) = 2$. Then θ has two lifts $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ with $\nu(\chi_1) = +1$ and $\nu(\chi_2) = -1$.

Projective modules

Theorem (M., Navarro 2015)

$T \backslash V$ contains an involution iff $P(S)$ affords a symplectic geometry.

Example (c.f. Gow-Willems)

$G = (C_3 : C_4) \wr C_2$ and $\dim(S) = 4$. Then $S \cong D_8$ and $T \cong D_8 : C_2$.
So $P(S)$ has a symplectic geometry.

Open Problem

Does the above theorem generalize to arbitrary finite groups?

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