# Beyond tame blocks

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# 1. Introduction

Aim: Study blocks in the wider context of symmetric algebras.

Finite type: cyclic defect groups $\longrightarrow$ Brauer tree algebras.

Tame type: dihedral / semidihedral/ quaternion defect groups

$$\longrightarrow ?\ ?$$

Previously: 'algebras of dihedral / semidihedral / quaternion type. Definition involve conditions, different in each case.

[E- Skowroński] 'weighted surface algebras':

• unified generalisations of (almost all) these algebras, hence of tame blocks.

This started with [S. Ladkani Jacobian algebras $J(Q, W)$].

Qu: do simples have period 4?

[E-Skowroński ] yes. Introduce 'weighted surface algebras'.

[Ladkani] introduces 'algebras of quasi-quaternion type'

## 2. Special case: the local algebras

$\Lambda_q = K\langle X, Y\rangle/(X^2 - YXY, Y^2 - XYX, (XY)^2X, (XY)^2Y)$

$\Lambda_{sd}$ Replace $Y^2 - XYX$ by $Y^2$.

$\Lambda_d$ Replace $X^2 - YXY$ by $X^2$ and $Y^2 - XYX$ by $Y^2$.

**Lemma** $\Lambda = \Lambda_q : \Omega^4(K) \cong K$.

*Proof*

**Lemma** $\Lambda = \Lambda_{sd}$ and $Ht(\Lambda) = \text{rad}\Lambda/\text{soc}\Lambda$. Then there is an AR sequence

$$0 \to V \to Ht(\Lambda) \to U \to 0$$

Hence $K$ belongs to a $D_\infty$ component.

**Lemma** $\Lambda = \Lambda_d$. Then $\text{Ht}(\Lambda)$ is a direct sum. Hence $K$ belongs to a $A_\infty^\infty$ component.

# 3. Triangulation quivers

**DEF** [ES/L] A triangulation quiver is a pair $(\mathcal{Q}, f)$ where $\mathcal{Q}$ is a quiver, and $f$ is a permutation of the arrows, such that
(i) For every vertex $i$, two arrows go in and two arrows come out.
(ii) For each arrow $\alpha$, we have $s(f\alpha) = t(\alpha)$.
(iii) $f^3 = 1$.

$(\mathcal{Q}, f)$ comes from a triangulation of a surface:

**DEF** (1) If $\alpha$ is an arrow then $\bar{\alpha}$ is the other arrow starting at $s(\alpha)$.

(2) Define a permutation $g$ of the arrows by

$$g(\alpha) := \overline{f\alpha}$$

Colouring: arrows on each cycle of $f$ have the same colour different cycles of $f$ have different colours. Then $g$ always changes colour.

# 5. Weighted surface algebras

$\mathcal{Q} =$ a triangulation quiver.

$(\alpha, g\alpha, \ldots)$ cycle of $g \quad \longrightarrow m_\alpha \geq 1$ and $c_\alpha \in K^*$ (constant on the cycle)

$A_\alpha := (\prod_i g^i \alpha)^{m_\alpha - 1}(\alpha g\alpha \ldots g^{n_\alpha - 1}\alpha)$.

**DEF** A weighted surface algebra is an algebra $A = K\mathcal{Q}/I$ where $I$ is given by the following relations:

$$\alpha \cdot f\alpha = c_{\bar{\alpha}} A_{\bar{\alpha}}, \quad \alpha \cdot f\alpha \cdot g(f\alpha)) = 0$$

(for each arrow $\alpha$)

**Structure of projectives**

# 7. Examples

**1.** Algebras of quaternion type fit into this.

**2.** Icosahedron, buckminsterfullerine,

**3** many others.

**Remark** Algebra depends on the orientation of triangles.

## 5. Generalizing semidihedral type

Let $T$ be a proper non-empty subset of the cycles of $f$. Degenerate the relations along each of these triangles. This defines an algebra $\Lambda_T$.

**Examples**

**1.** $\Lambda_{sd}$

**2.** other:

This has at least one simple module in a $D_\infty$ component. It may have simples in $A_\infty^\infty$ component, It may also have simple modules of period four.

These all were conditions for the original semidihedral type algebra.

## 6. Generalizing dihedral type

Degenerate all relations of the triangles. This gives a special biserial algebra. All simples lie in $A_{\infty}^{\infty}$ components.

**Lemma** All modules $\alpha\Lambda_d$ have $\Omega$-period 3.

This was required for the original 'dihedral type' algebra.

# 8. Results

• Weighted surface algebras are symmetric The Cartan matrix is non-singular only for quaternion type.

• Weighted surface algebras are tame: They degenerate to special biserial algebras.

**Theorem** Assume $A$ is a weighted surface algebra and $S_i$ is simple. Then $S_i$ has $\Omega$-period four.

Same proof as for the local algebra.

Improving the Theorem:

**Theorem** $A$ is periodic as a bimodule, with period 4.

If $\mathcal{Q}$ has loops and $K$ has characteristic 2 then $A$ has socle deformations $A_q$. These are also periodic as bimodules.

[The proof is very similar to the proof for the quaternion type algebras.]

## 8. Open problems

We do not know for the original quaternion type:

(1) precisely which algebras of quaternion type occur as blocks.

(2) whether there could be $\infty$ many blocks up to $\sim_M$.

(3) a classification of indecomposable modules.

**Feasible to do:**

Let $A = KG$, $G =$ generalised quaternion group of order $2^n$. Then $G < D$ where $D$ is semidihedral of order $2^{n+1}$.

The indecomposable modules for $KD$ are classified [W. Crawley-Boevey]

If $M$ is an indecomposable $KG$-module then $M\uparrow^D$ is indecomposable.

**Qu.** Can one identify which indecomposable $KD$-modules occur in this way?