# Donovan's conjecture


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# Notation

$\mathcal{O}$- complete discrete valuation ring (characteristic 0)
$k$ - residue field, algebraically closed, characteristic $p$
$G$ - finite group
Consider blocks $B$ of $kG$ (and $\mathcal{O}G$), defect group $D$

- Interested in the module categories $\mathrm{mod}(kG)$ and $\mathrm{mod}(\mathcal{O}G)$, f.g. modules.
- Suffices to consider $\mathrm{mod}(B)$ for blocks $B$.
- The (Morita) equivalence class of $\mathrm{mod}(B)$ is determined by the isomorphism class of a basic algebra $eBe$ of $B$.
- Here $e$ is a sum of primitive idempotents of $B$, one for each isomorphism class of projective indecomposable $B$-modules.

# Why study Morita equivalences?

Most invariants we're interested in are Morita invariant.

E.g., the Cartan matrix (up to rearrangement), Loewy length, $Z(B)$, number of irreducible characters associated to $B$.

**However** Morita equivalent blocks not known to have isomorphic defect groups (or the same fusion).

This doesn't fit well with the ethos of $p$-local determination.

**Aside on subject of centres -** Theorem (Schwabrow) Let $G = Ree(q)$, $q = 3^{2m+1}$, $P \in Syl_3(G)$. Then $Z(B_0(kG)) \ncong Z(B_0(kN_G(P)))$.

# Source algebras

Due to Puig. In some sense an answer to $p$-local issue.

A source idempotent $f$ of $B$ is a primitive idem. of $B^D$ s.t $Br_D(f) \neq 0$.

$fBf$ is a source algebra for $B$. Blocks are source algebra equivalent if their source algebras are isomorphic as interior $D$-algebras.
Properties:

- $fBf$ is Morita equivalent to $B$ ($fe = e$, where $e$ is some basic algebra idempotent).
- $D$ embeds in $fBf$ via $fDf$, so source algebra equivalence preserves defect groups.
- Source algebra equivalence preserves fusion, vertices and sources, generalized decomposition numbers.
- Source algebra equivalence over $k$ lifts to source algebra equivalence over $\mathcal{O}$ (because $fBf$ is a $D \times D$-permutation module).
- Source algebra equivalence lifts to central extension by $p$-groups.
- Source algebras are rigid, in that there are only finitely many isomorphism types of a given dimension (Puig, unpublished)

# More problems with Morita equivalences

Following Kessar: Let $\sigma$ be a field automorphism of $k$.

$\sigma$ extends to a ring automorphism of $kG$ via action on coefficients.

Blocks are permuted by $\sigma$.

$B$ is *Galois conjugate* to $B^\sigma$.

Galois conjugate blocks are isotypic.

Benson and Kessar (2007): There are Galois conjugate blocks which are not Morita equivalent.

The examples constructed are solvable, with normal abelian defect groups.

In all known examples $B^{\sigma^2}$ is Morita equivalent to $B$.

# Finiteness conjectures

All of the following were orginally stated as questions.

**Conjecture (from Brauer's Problem 22)**

*Fix a p-group $P$. Then $\exists\ c = c(P)$ such that for all finite groups $G$ and blocks $B$ of $G$ with defect group $\cong$ to $P$, the entries of $C_B$ are all at most $c$.*

**Conjecture (Donovan)**

*Fix a p-group $P$. Then there are only a finite number of Morita equivalence classes of blocks of finite groups with defect groups $\cong$ to $P$.*

This can be asked for blocks with respect to $k$ or to $\mathcal{O}$.

**Conjecture (Puig)**

*Fix a p-group $P$. Then there are only a finite number of source equivalence classes of blocks of finite groups with defect groups $\cong$ to $P$.*

Clear that Puig $\Rightarrow$ Donovan $\Rightarrow$ Brauer.

Let $\sigma$ be the Frobenius map on $k$. For a block $B$ of $kG$, let $m(G,B)$ be the number of Morita equivalence classes amongst $\{B^{\sigma^n} : n \in \mathbb{N} \cup \{0\}\}$.

### Conjecture (Kessar)

*Fix a p-group $P$. There is $m \in \mathbb{N}$ such that for all blocks with defect group $\cong$ to $P$, we have $m(G,B) \leq m$.*

### Theorem (Kessar ('04))

*For a fixed $P$, Brauer's conjecture and the above conjecture are equivalent to Donovan's.*

Based on the realizability of a basic algebra over a finite subfield $\mathbb{F}$ of $k$, i.e., existence of an $\mathbb{F}$-algebra $A$ such that $ekGe \cong k \otimes_{\mathbb{F}} A$.

### Corollary (Kessar)

*For a fixed p-group $P$, Donovan's conjecture for principal blocks is equivalent to Brauer's.*

# Approaches to Morita equivalence

Three main strands.

- Understand the basic algebras. E.g., work of Erdmann on tame type - more on next slide.
- Reduction theorems and the classification - more later
- Work within classes of simple groups. Motivated by work of Scopes on symmetric groups (1991). Much work by Hiss and Kessar on unipotent blocks classicial groups (2000, 2005), and alternating group (Kessar 2001).

# Finite and tame type

**Cyclic defect groups:** Donovan's conjecture (over $k$) known by Dade, Janusz, Kupisch (1960's).

Morita equivalence classes determined by Brauer tree.

Puig's conjecture by Linckelmann (1996)

**Tame type:** Here $p = 2$ and $D$ is generalized quaternion, dihedral or semidiedral.

Erdmann described the basic algebras (over $k$) as occuring in series of algebras via Auslander-Reiten theory, leading to Donovan's conjecture in all tame cases except $D$ qeneralized quaternion with $l(B) = 2$.

E.g., (i) $P = D_8$. Four Morita equivalence classes (w.r.t. $k$), represented by the principal blocks of $kP$, $kA_7$, $kPSL_2(7)$, $kPSL_2(9)$. (Erdmann 1987).

(ii) $P = Q_8$. Three Morita equivalence classes (w.r.t. $k$), represented by the principal blocks of $kP$, $kSL_2(3)$, $kSL_2(5)$. (Erdmann 1988).

# Tame type (continued)

**Klein four groups:** Source algebras shown to lie within three infinite families by Linckelmann (1994).

By Craven, E, Kessar and Linckelmann (2011), using CFSG, only one from each family occurs. So equivalence class representatives are the principal blocks of:

$\mathcal{O}D$, $\mathcal{O}A_4$ and $\mathcal{O}A_5$.

Note on proof: Not achieved by straight reduction.

Showed that every such block has a simple trivial source module.

# Blocks and normal subgroups

Necessary for reductions to quasisimple groups.

**Theorem (Külshammer 1995)**

*Fix $P$. In order to verify Donovan's conjecture for $P$ (w.r.t. $k$), it suffices to consider groups $G$ with $G = \langle D^g : g \in G \rangle$, where $D$ is a defect group.*

**Ingredients:** (i) Suppose there is $N \lhd G$ with $D \leq N$.

$B$ is Morita equivalent to an algebra $Y$ which is a crossed product of a basic algebra of a block of required type with defect group $D$ with a finite $p'$-group of order at most $|\mathrm{Out}(D)|^2$.

(ii) Finiteness result on crossed products with $p'$-groups.

# A note on nilpotent blocks

$B$ is nilpotent if $N_G(Q, b_Q)/C_G(Q)$ is a $p$-group for each $Q \leq D$ and each $B$-subpair $(Q, b_Q)$.

Nilpotent blocks are Morita equivalent to $kD$, where $D$ is a defect group (Broué-Puig).

**Example**: If $D$ is a cyclic 2-group, then $N_G(Q)/C_G(Q)$ is a 2-group for all $Q \leq D$, and so $B$ is automatically nilpotent.

A nilpotent block extends its influence to blocks it covers.

**Theorem (Puig, 2010)**

*Let $N \lhd G$ and $b$ be a block of $N$ covered by a nilpotent block $B$ of $G$. Then $b$ is Morita equivalent to a block of a subgroup of $N_N(Q)$, where $Q$ is a defect group for $b$.*

# Külshammer-Puig theory

Let $N \lhd G$ and $B$ be a block of $G$ covering a block $b$ of $N$.

Let $D$ be a defect group for $B$ such that $D \cap N$ is a defect group for $b$.

By Fong-Reynolds, $B$ is Morita equivalent to a block of $Stab_G(b)$, which we may assume also has defect group $D$.

Hence may assume all blocks of normal subgroups covered by $B$ are $G$-stable.

Now assume $b$ nilpotent.

Külshammer-Puig (1990): $B$ Morita equivalent to a block of a $p'$-central extension of $L$ with $D \cap N \lhd L$ and $L/(D \cap N) \cong G/N$. Defect group preserved under Morita equivalence.

Another result very useful for reductions:

**Theorem (Koshitani-Külshammer, 1996)**

*Let $G$ be a finite group and let $B$ be a block of $kG$ with defect group $D$. Suppose $N \trianglelefteq G$ with $G = ND$ and $D = (D \cap N) \times Q$ for some $Q$. If $B$ covers a $G$-stable block $b$ of $kN$, then $B$ is Morita equivalent to a block $C$ of $k(N \times Q)$ with defect group $D$.*

If $D$ is elementary abelian, then the splitting condition always holds when $G = ND$.

Hence in proving Donovan's conjecture for elementary abelian groups, may assume $O^p(G) = G$.

**Note:** The above result holds modulo $O_p(Z(G))$, where we still require that the defect group is abelian.

# Known general reductions

Düvel (2004) showed:

(i) Reduction of Donovan for principal blocks to simple groups ($P$-abelian)

(ii) Reduction of Donovan to decorations of central products of quasisimples ($P$-abelian)

(iii) Reduction of Brauer's conjecture to quasisimples (hence Donovan for principal blocks).

# Blocks with abelian 2-defect groups

**Theorem (Eaton-Kessar-Külshammer-Sambale, 2013)**

*Suppose that $G$ is a quasi-simple group and $B$ is a 2-block of $G$ with abelian defect group $D$. Then one of the following holds.*

- *$G/Z(G)$ is one of $PSL_2(2^a)$, ${}^2G_2(q)$ ($q = 3^{2m+1}$), or $J_1$. Here $B$ is the principal block and $D$ is elementary abelian.*
- *$G$ is $Co_3$ and $B$ is a non-principal block, $D \cong C_2 \times C_2 \times C_2$.*
- *$B$ is Morita equivalent to a block covered by a nilpotent block.*
- *$B$ is Morita equivalent to a block $C$ of a group $L = L_0 \times L_1$, where $L_0$ is abelian and $C$ covers a block of $L_1$ with Klein four defect groups.*

Using Düvel's reduction of Brauer's Conjecture to quasisimple groups, we have:

**Theorem (EKKS, 2013)**

*Fix an abelian 2-group $P$. Then $\exists\ c = c(P)$ such that for all finite groups $G$ and all blocks $B$ of $G$ with defect group $P$, the entries of $C_B$ are all at most $c$.*

**Theorem (EKKS, 2013)**

*Let $P$ be an elementary abelian 2-group. Then Donovan's conjecture holds for $P$.*

**Theorem (EKKS)**

*Let $P \cong C_{2^n} \times C_{2^n}$. If $n > 1$, then there are two Morita equivalence classes of blocks with defect group $P$, with representatives $kP$ and $k(P \rtimes C_3)$.*

# $C_2 \times C_2 \times C_2$

Using the EKKS classification of 2-blocks with abelian defect groups and work by many others:

**Theorem (E 2014)**

*Let $P \cong C_2 \times C_2 \times C_2$. There are eight Morita equivalence classes of blocks with defect group $P$, representatives the principal blocks of the following:*
*(i) $\mathcal{O}D$;*
*(ii) $\mathcal{O}D \rtimes C_3$;*
*(iii) $\mathcal{O}C_2 \times A_5$ (inertial quotient is $C_3$);*
*(iv) $\mathcal{O}D \rtimes C_7$;*
*(v) $\mathcal{O}SL_2(8)$ (inertial quotient is $C_7$);*
*(vi) $\mathcal{O}D \rtimes (C_7 \rtimes C_3)$;*
*(vii) $\mathcal{O}J_1$ (inertial quotient is $C_7 \rtimes C_3$);*
*(viii) $\mathcal{O}\ {}^2G_2(3)$ (inertial quotient is $C_7 \rtimes C_3$), note ${}^2G_2(3) \cong \mathrm{Aut}(SL_2(8))$.*

W.r.t. $k$, this completes the classification of Morita equivalence classes of 2-blocks of defect at most 3.

# Abelian 3-groups

**Theorem (Koshitani 2003)**

*Donovan's conjecture holds for principal blocks for abelian 3-groups, w.r.t. $k$.*