Counting Simple Modules: A conjectured inequality

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As usual, $(\mathbb{K}, R, \mathbb{F})$ is a p-modular system for a finite group G. We let $k(B)$ denote the number of ordinary irreducible characters in the block B, and $\ell(B)$ denote the number of irreducible Brauer characters in B. In his famous list of problems, R. Brauer asked whether we always have $k(B) \leq |D|$ when B is a block of RG with defect group D. This question remains open at present, although much evidence has been accumulated towards an affirmative answer. In particular, it is known (by the work of many authors) to be true for all blocks of p-solvable groups.

One difficulty with proving the inequality in general is that it does not seem to succumb easily to standard Clifford theoretic reductions.

One motivation for the conjecture we make on $\ell(B)$ is that it may provide some useful inductive information for Brauer's question (though we believe that the conjecture has its own intrinsic interest).

The sectional rank of a finite group X, denoted $r(X)$, is the maximum number of generators of any section of X, or equivalently, the maximum number of generators of any subgroup of X. When X is a p-group, this is also the maximum rank of an elementary Abelian section of X. When B is a p-block with defect group D, we set $s(B) = r(D)$.

Main Conjecture: Let B be a block of RG with defect group D. Then we have
$\ell(B) \leq p^{s(B)}$.

This conjecture is easily checked for blocks with cyclic defect group and for tame blocks. The next evidence we offer towards the main conjecture is:

Theorem: *The main conjecture is true for all blocks of p-solvable groups, and in that case the inequality is strict for blocks of positive defect.*

Outline Proof: By the usual Fong reductions, we need to consider a p-solvable group G of order divisible by p with $O_{p'}(G) \leq Z(G)$.

Other reductions allow us to assume that $Z(G)$ is a p'-group and that $U = O_p(G)$ is elementary Abelian. Now there is a linear character λ of $Z = Z(G)$ such that $\ell(B)$ is the number of projective indecomposable RG-modules whose characters lie over λ.

Again by work of Fong, any character θ of such a projective indecomposable module RG module is induced from an irreducible character (which must lie over λ) of a Hall p'-subgroup H of G. In particular, $\ell(B) \leq k(H, \lambda) < k(UH, \lambda)$. By the solution of the coprime $k(GV)$-problem, we have $k(UH, \lambda) \leq |U|$, so $\ell(B) < |U|$, which suffices to complete the proof.

Now we make a few remarks (without proof) on the structure of a minimal counterexample G and a putative block B for which the inequality fails.

It is clear that the block B is quasiprimitive, and that $G = O^p(G)$. Furthermore, $Z(G) = O_{p'}(G) \leq \Phi(G)$ and $O_p(G)$ is elementary Abelian.

It is now natural to divide the analysis into two cases:
The "constrained case" : In this case, $O^p(F^*(G)) = Z(G)$. Given the above restrictions, this is equivalent to assuming that G has no components.

The "non-constrained" case: In this case, $E(G) \neq 1$.

Remarks on the constrained case: The constrained case reduces fairly easily to the case when B is the principal block, and G is a semidirect product of the form $G = VH$ where V is a faithful irreducible $\mathbb{F}_p H$-module and H is a finite group of order divisible by p. In that case, it would be sufficient (though maybe not necessary) to prove that $\ell(H) < |V|$. It is not always true that $k(VH) \leq |V|$ though Guralnick and Maróti and Guralnick and Tiep have established that stronger inequality in many cases when V is a primitive module.

Remarks on the non-constrained case: Here, our first goal is to understand the case when $F^*(G)$ is quasisimple, or at least when G itself is quasisimple.

Established cases (to date): The conjecture is true (with strict inequality) for all blocks of symmetric and alternating groups, and their covering groups. The proof here makes much use of work of J.B. Olsson on
Brauer's $k(B) \leq |D|$ question.

The conjecture is true (with strict inequality) for all blocks of quasisimple Lie type characteristic p groups (and, in fact still holds if we extend by a field,graph, or graph-field automorphism).

The conjecture holds when $F^*(G)/Z(G)$ is a covering group of a sporadic simple group, or the Tits group. The inequality is strict unless perhaps $p = 3$ and B has extraspecial defect groups of order 27 (and such exceptions do occur).

When p is not the defining characteristic

The conjecture holds (with strict inequality) for all blocks (for all primes) of exceptional covering groups of finite simple groups of Lie type.

The conjecture holds for all blocks of quasisimple groups of Lie type ${}^2B_2, {}^2G_2, G_2, {}^3D_4$ or ${}^2F_4$, and the inequality is strict unless the block B is the principal 3-block of ${}^2F_4(q^2)^{(\prime)}$ for $q^2 \equiv 2$ or 5 (mod 9).

The conjecture holds whenever B is a p-block of $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$ whenever $q = r^m$ is a power of a prime $r \neq p$. This makes heavy use of work of Fong and Srinivasan.

The conjecture holds when B is a unipotent p-block of quasimple group of classical Lie type in characteristic $r \neq p$.

The conjecture holds for all unipotent p-blocks of quasisimple groups G of exceptional groups of Lie type in characteristic $r \neq p$ such that p is good for G.

Further information is available when p is not the defining characteristic, and work of Bonnafé and Rouquier should be helpful. Note though that we can only exploit Morita (or derived) equivalences when they preserve the sectional rank of the defect group.

Relationship with other conjectures: We are unsure whether there are implications in either direction for general blocks between the main conjecture here and Alperin's weight conjecture, though AWC would appear to be more precise than the main conjecture here.

The main conjecture holds (with strict inequality) whenever B is a p-block with non-trivial Abelian defect group satisfying either Broué's Abelian defect group conjecture or Alperin's weight conjecture. This makes use of Morita equivalences constructed by Külshammer and Reynolds, as well as the solution of the
$k(B) \leq |D|$ problem for p-solvable groups.

A consequence: If the main conjecture holds for all p-blocks, then whenever B is a p-block with defect group D, we would have $k(B) \leq p^{s(B)} k_G(D,b)$, where $k_G(D,b)$ is the number of Brauer elements (up to G-conjugacy) contained in the maximal B-subpair (D,b). In particular, this yields, after some more work, $k(B) \leq p^{r(D)}(k(D)-1)$. The general bound can be less than $|D|$. It is usually (but not always) much less than the Brauer-Feit bound $\frac{|D|^2}{p^2}$, which is the best bound currently known for $k(B)$ for a general p-block B. The second bound also relates to a question of Malle and Navarro.

(Late) Happy Birthday Burkhard!
Next objective $|SL_2(5)|$ (exclamation mark omitted as factorial too ambitious).